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Improvement of substructuring reduction technique for large eigenproblems using an efficient dynamic condensation method

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Abstract

An accelerated substructuring reduction procedure for the iterated improved reduced system (IIRS) method is proposed. The iterated IIRS method can be combined with a substructuring scheme to provide an efficient methodology for large-scale eigenvalue problems. Not only can it reduce eigenvalue analysis errors through successive iterations, but the accuracy of the eigenanalysis is not sensitive to the selected master degrees of freedom. In practical structural eigenproblems, reducing the number of iterations can save a great deal of computation cost. The present substructuring technique modifies the iterative form of the transformation matrices in each substructure to achieve faster convergence. Applications of the present method to two numerical examples demonstrate that the proposed method can obtain lower eigensolutions of structures more accurately and efficiently, as compared with those of the current substructuring technique.

Keywords: Dynamic condensation method; Iterated IRS (IIRS) method; Substructuring technique

1. Introduction

Modern structural dynamics using finite element methods requires computational models having a very large number of degrees of freedom if structural engineers are to accurately evaluate the response of structures under detailed models. Eigenvalue problems of such structures need a large amount of computing time. Although modern supercomputers can solve more than several million degrees of freedom problems, the analysis cost is very high and they are not easily accessible by most engineers who work for daily design and analysis jobs. Therefore, many researchers have been interested in solving large-scale eigenvalue problems with limited computer storage and speed.

One of the ways to resolve these problems is to reduce the size of the problem. This way involves truncating the higher modes from the given full system or eliminating the unimportant degrees of freedom. These researches on constructing reduced models have proceeded in two different ways. One is a reduced-order method which constructs a reduced system with a few modes dominating the response of a structure. The other is a condensation method in which the reduced matrices are constructed with the master degrees of freedom by transformation matrix. The former has the advantage of simplicity in constructing a reduced system and does not require much computational resources. But the truncation of higher modes leads to increase the errors of eigenvalues and eigenvectors. On the other hand, the latter can calculate more accurate eigenproperties than the former but this method requires much computational cost because of the construction of transformation matrix. Therefore, the latter method can be computationally efficient reduction techniques if the transformation matrix is constructed without consuming much com-

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putational cost.

For the last several decades various approximation techniques have been developed to calculate eigenproperties by the dynamic condensation method. The condensation technique was first proposed by Guyan [1] and Irons [2] in 1965. These methods involve elimination of the degrees of freedom which do not give any significant influence on the solution field. But the accuracy of those methods was very low because the inertia effect was not considered properly when constructing the condensation matrices. O'Callahan [3]improved Guyan's method by considering the first-order approximation terms in the transformation formula of the slave degrees of freedom. Although O'Callahan's method provides a better result than that of Guyan, it may have a non-positive definite mass matrix by the improper selection of the master degrees of freedom. Godis [4] generated the transformation for the standard IRS (Improved Reduced System) method by using a binomial series expansion in approximating the eigenvalue term. An iterative dynamic condensation method was proposed by Suarez and Singh [5]. In this method the eigensolution was obtained by using the orthogonality conditions of the eigenvectors. Friswell, Garvey and Penny [6] proposed an iterated IRS (IIRS) technique, and later proved the convergence [7].

Recently, an iterative method for nonclassically damped systems was proposed by Qu [8-11]. Qu presented various condensation methods for nonclassically damped systems defined in displacement space and state space. Most recently, Xia and Lin [12] proposed an improved dynamic condensation technique by modifying the iterative transformation matrix and accelerated the convergence. Through this technique, a more accurate and efficient lowest eigensolution of structures was obtained in comparison with the IIRS method. Kim and Cho [14] proposed a two-level condensation scheme for an undamped structural system and calculated the sensitivity from the reduced system. In this scheme the reduced matrices are constructed by the well-selected primary degrees of freedom through the element level energy estimation [13].

However, although these condensation techniques can reduce the size of the model drastically, it takes a large amount of computing time for the construction of the reduced system when the problem has a large number of degrees of freedom over several hundred thousands. One of the ways to overcome this problem is to apply a substructuring scheme. In static and dynamic problems, if the whole structure can be separated into substructures, then it can be solved more readily with limited memory.

Crag-Bampton [15] employed component mode synthesis for dynamic analysis. In the 1990's, Aminpour et al. [16] performed a coupled analysis with the independent sub-domains by hybrid interface formulation. N.Bouhaddi and R.Filld [17, 18] proposed the dynamic substructuring method using Guyan condensation method based on the important degrees of freedom in the matching system. Kim and Cho [19] developed three-type sub-domain schemes by combining a two-level condensation scheme with a substructuring scheme.

More recently, Choi and Cho [20, 21] proposed the IIRS method combined with substructuring scheme for undamped structural systems and for nonclassically damped systems. This method can calculate highly accurate eigenproperties from repeatedly updated condensed matrices without consuming expensive computational cost for large structures.

The objective of this study is to accelerate the convergence speed of the substructuring scheme which is combined with the iterated IRS method. This can be achieved by modifying the iterative formula of the transformation matrix. Two numerical examples are provided to demonstrate the convergence and efficiency of the newly developed algorithm.

2. Substructuring for IIRS method

The dynamic equilibrium of an N-degree-of-freedom system can be expressed in a matrix form as

$$\mathbf{M}\ddot{X}(t)+\mathbf{C}\dot{X}(t)+\mathbf{K}X(t)=\mathbf{f}(t) \tag{1}$$

where the mass matrix $\mathbf{M}$, damping matrix $\mathbf{C}$, and stiffness matrix $\mathbf{K}$ are assumed to be positive definite, positive semidefinite, and positive semidefinite, respectively. The corresponding eigenvalue problem for undamped system may be written in displacement space as

$$\mathbf{K\Phi}=\mathbf{M\Phi}\Lambda \tag{2}$$

where $\mathbf{\Phi}$ is the eigenvector, representing the vibrating mode, corresponding to the eigenvalue Λ .

For developing a basic formulation of substructuring, the whole system is just divided into two substructures and the system matrices are constructed in each substructure. In the following equations, the subscripts "m" and "s" represent the master and slave degrees of freedom, respectively. The eigenvalue problem for substructure one can be expressed in a partitioned form as

$$\begin{bmatrix} \mathbf{K}_{ss}^{(1)} & \mathbf{K}_{sm}^{(1)} \\ \mathbf{K}_{ms}^{(1)} & \mathbf{K}_{mm}^{(1)} \end{bmatrix} \begin{bmatrix} \mathbf{\Phi}_{sm}^{(1)} \\ \mathbf{\Phi}_{mm} \end{bmatrix} = \begin{bmatrix} \mathbf{M}_{ss}^{(1)} & \mathbf{M}_{sm}^{(1)} \\ \mathbf{M}_{ms}^{(1)} & \mathbf{M}_{mm}^{(1)} \end{bmatrix} \begin{bmatrix} \mathbf{\Phi}_{sm}^{(1)} \\ \mathbf{\Phi}_{mm} \end{bmatrix} \Lambda_{mm} \tag{3a}$$

and the eigenvalue problem for substructure two can also be described in a separated form as

$$\begin{bmatrix} \mathbf{K}_{mm}^{(2)} & \mathbf{K}_{ms}^{(2)} \\ \mathbf{K}_{sm}^{(2)} & \mathbf{K}_{ss}^{(2)} \end{bmatrix} \begin{bmatrix} \mathbf{\Phi}_{mm} \\ \mathbf{\Phi}_{sm}^{(2)} \end{bmatrix} = \begin{bmatrix} \mathbf{M}_{mm}^{(2)} & \mathbf{M}_{ms}^{(2)} \\ \mathbf{M}_{sm}^{(2)} & \mathbf{M}_{ss}^{(2)} \end{bmatrix} \begin{bmatrix} \mathbf{\Phi}_{mm} \\ \mathbf{\Phi}_{sm}^{(2)} \end{bmatrix} \Lambda_{mm} \tag{3b}$$

In Eq. (3a) and Eq. (3b), the stiffness matrix and mass matrix can be assembled into one global system as

$$\begin{bmatrix} \mathbf{K}_{ss}^{(1)} & \mathbf{K}_{sm}^{(1)} & \\ \mathbf{K}_{ms}^{(1)} & \mathbf{K}_{mm} & \mathbf{K}_{ms}^{(2)} \\ & \mathbf{K}_{sm}^{(2)} & \mathbf{K}_{ss}^{(2)} \end{bmatrix} \begin{bmatrix} \mathbf{\Phi}_{sm}^{(1)} \\ \mathbf{\Phi}_{mm} \\ \mathbf{\Phi}_{sm}^{(2)} \end{bmatrix} = \begin{bmatrix} \mathbf{M}_{ss}^{(1)} & \mathbf{M}_{sm}^{(1)} & \\ \mathbf{M}_{ms}^{(1)} & \mathbf{M}_{mm} & \mathbf{M}_{ms}^{(2)} \\ & \mathbf{M}_{sm}^{(2)} & \mathbf{M}_{ss}^{(2)} \end{bmatrix} \begin{bmatrix} \mathbf{\Phi}_{sm}^{(1)} \\ \mathbf{\Phi}_{mm} \\ \mathbf{\Phi}_{sm}^{(2)} \end{bmatrix} \Lambda_{mm} \tag{4}$$

where $\mathbf{K}_{mm} = \mathbf{K}_{mm}^{(1)} + \mathbf{K}_{mm}^{(2)}$ and $\mathbf{M}_{mm} = \mathbf{M}_{mm}^{(1)} + \mathbf{M}_{mm}^{(2)}$ including the interface degrees of freedom which connect each substructure. To eliminate the slave degrees of freedom field in each substructure, we employ the first and the third rows of Eq. (4) as

$$\begin{aligned} \mathbf{K}_{ss}^{(1)}\mathbf{\Phi}_{sm}^{(1)} + \mathbf{K}_{sm}^{(1)}\mathbf{\Phi}_{mm} &= \left(\mathbf{M}_{ss}^{(1)}\mathbf{\Phi}_{sm}^{(1)} + \mathbf{M}_{sm}^{(1)}\mathbf{\Phi}_{mm}\right)\Lambda_{mm} \\ \mathbf{K}_{sm}^{(2)}\mathbf{\Phi}_{mm} + \mathbf{K}_{ss}^{(2)}\mathbf{\Phi}_{sm}^{(2)} &= \left(\mathbf{M}_{sm}^{(2)}\mathbf{\Phi}_{mm} + \mathbf{M}_{ss}^{(2)}\mathbf{\Phi}_{sm}^{(2)}\right)\Lambda_{mm} \end{aligned} \tag{5}$$

Through Eq. (5) the transformation relation of the master degrees of freedom field and the slave degrees of freedom field can be obtained. Rearranging Eq. (5) for the slave degrees of freedom field,

$$\begin{aligned} \mathbf{\Phi}_{sm}^{(1)} &= -\left(\mathbf{K}_{ss}^{(1)}\right)^{-1}\mathbf{K}_{sm}^{(1)}\mathbf{\Phi}_{mm} \\ &\quad + \left(\mathbf{K}_{ss}^{(1)}\right)^{-1}\left(\mathbf{M}_{sm}^{(1)}\mathbf{\Phi}_{mm} + \mathbf{M}_{ss}^{(1)}\mathbf{\Phi}_{sm}^{(1)}\right)\Lambda_{mm} \\ \mathbf{\Phi}_{sm}^{(2)} &= -\left(\mathbf{K}_{ss}^{(2)}\right)^{-1}\mathbf{K}_{sm}^{(2)}\mathbf{\Phi}_{mm} \\ &\quad + \left(\mathbf{K}_{ss}^{(2)}\right)^{-1}\left(\mathbf{M}_{sm}^{(2)}\mathbf{\Phi}_{mm} + \mathbf{M}_{ss}^{(2)}\mathbf{\Phi}_{sm}^{(2)}\right)\Lambda_{mm} \end{aligned} \tag{6}$$

According to the definition of the transformation matrices in each subsystem, that are,

$$\begin{aligned} \mathbf{\Phi}_{sm}^{(1)} &= \mathbf{t}_{(1)}\mathbf{\Phi}_{mm} \\ \mathbf{\Phi}_{sm}^{(2)} &= \mathbf{t}_{(2)}\mathbf{\Phi}_{mm} \end{aligned} \tag{7}$$

Substituting Eq. (7) into Eq. (6) and rearranging the result yields

$$\begin{aligned} \mathbf{t}_{(1)} &= -\left(\mathbf{K}_{ss}^{(1)}\right)^{-1}\mathbf{K}_{sm}^{(1)} + \left(\mathbf{K}_{ss}^{(1)}\right)^{-1} \\ &\quad \left(\mathbf{M}_{sm}^{(1)} + \mathbf{M}_{ss}^{(1)}\mathbf{t}_{(1)}\right)\mathbf{\Phi}_{mm}\Lambda_{mm}\mathbf{\Phi}_{mm}^{-1} \\ \mathbf{t}_{(2)} &= -\left(\mathbf{K}_{ss}^{(2)}\right)^{-1}\mathbf{K}_{sm}^{(2)} + \left(\mathbf{K}_{ss}^{(2)}\right)^{-1} \\ &\quad \left(\mathbf{M}_{sm}^{(2)} + \mathbf{M}_{ss}^{(2)}\mathbf{t}_{(2)}\right)\mathbf{\Phi}_{mm}\Lambda_{mm}\mathbf{\Phi}_{mm}^{-1} \end{aligned} \tag{8}$$

From Eq. (8), we get two transformation matrices. By these two transformation matrices, the whole assembled system can be reduced to the one with only master degrees of freedom field as

$$\begin{bmatrix} \mathbf{\Phi}_{sm}^{(1)} \\ \mathbf{\Phi}_{mm} \\ \mathbf{\Phi}_{sm}^{(2)} \end{bmatrix} = \begin{bmatrix} \mathbf{t}_{(1)} \\ \mathbf{I}_{mm} \\ \mathbf{t}_{(2)} \end{bmatrix} \mathbf{\Phi}_{mm} = \mathbf{T}\mathbf{\Phi}_{mm} \tag{9}$$

where $\mathbf{I}$ is the unit matrix of size $m{\times}m$ and $\mathbf{T}$ is the transformation matrix between $\mathbf{\Phi}_{mm}$ and $\mathbf{\Phi}_{sm}$ in the whole system. Substituting Eq. (9) into Eq. (4) and premultiplying $\mathbf{T}^{\mathrm{T}}$ on the left of the equation, we can obtain the reduced system matrices as

$$\begin{aligned} \mathbf{K}_R &= \mathbf{t}_{(1)}^{\mathrm{T}}\mathbf{K}_{ss}^{(1)}\mathbf{t}_{(1)} + \mathbf{K}_{ms}^{(1)}\mathbf{t}_{(1)} + \mathbf{t}_{(1)}^{\mathrm{T}}\mathbf{K}_{sm}^{(1)} \\ &\quad + \mathbf{K}_{mm} + \mathbf{t}_{(2)}^{\mathrm{T}}\mathbf{K}_{sm}^{(2)} + \mathbf{K}_{ms}^{(2)}\mathbf{t}_{(2)} + \mathbf{t}_{(2)}^{\mathrm{T}}\mathbf{K}_{ss}^{(2)}\mathbf{t}_{(2)} \\ \mathbf{M}_R &= \mathbf{t}_{(1)}^{\mathrm{T}}\mathbf{M}_{ss}^{(1)}\mathbf{t}_{(1)} + \mathbf{M}_{ms}^{(1)}\mathbf{t}_{(1)} + \mathbf{t}_{(1)}^{\mathrm{T}}\mathbf{M}_{sm}^{(1)} + \mathbf{M}_{mm} \\ &\quad + \mathbf{t}_{(2)}^{\mathrm{T}}\mathbf{M}_{sm}^{(2)} + \mathbf{M}_{ms}^{(2)}\mathbf{t}_{(2)} + \mathbf{t}_{(2)}^{\mathrm{T}}\mathbf{M}_{ss}^{(2)}\mathbf{t}_{(2)} \end{aligned} \tag{10}$$

Through above equation the basic substructuring reduction procedure is derived. However, the transformation matrix in each substructure did not defined completely. In Eq. (10), we can construct a reduced eigenproblem of size m degrees of freedom as

$$\mathbf{K}_R\mathbf{\Phi}_{mm} = \mathbf{M}_R\mathbf{\Phi}_{mm}\Lambda_{mm} \quad (11)$$

From Eq. (11), we get

$$\mathbf{\Phi}_{mm}\Lambda_{mm}\mathbf{\Phi}_{mm}^{-1} = \mathbf{M}_R^{-1}\mathbf{K}_R \quad (12)$$

Substituting Eq. (12) into Eq. (8), we get two transformation matrices as

$$\begin{aligned}\mathbf{t}_{(1)} &= -\left(\mathbf{K}_{ss}^{(1)}\right)^{-1}\mathbf{K}_{sm}^{(1)} + \left(\mathbf{K}_{ss}^{(1)}\right)^{-1} \\ &\quad \left(\mathbf{M}_{sm}^{(1)} + \mathbf{M}_{ss}^{(1)}\mathbf{t}_{(1)}\right)\mathbf{M}_R^{-1}\mathbf{K}_R \\ \mathbf{t}_{(2)} &= -\left(\mathbf{K}_{ss}^{(2)}\right)^{-1}\mathbf{K}_{sm}^{(2)} + \left(\mathbf{K}_{ss}^{(2)}\right)^{-1} \\ &\quad \left(\mathbf{M}_{sm}^{(2)} + \mathbf{M}_{ss}^{(2)}\mathbf{t}_{(2)}\right)\mathbf{M}_R^{-1}\mathbf{K}_R\end{aligned} \quad (13)$$

Since the above equations are nonlinear, it's not easy to solve them directly; thus, they have to be solved by an iterative manner. The detailed iterative procedure is shown in the Appendix. From Eq. (A9 ~ A10), the lowest m eigenvalues and the associated eigenvector after (k-1)th iteration are estimated by solving the generalized eigenproblem of the reduced system:

$$\mathbf{K}_R^{(k)}\mathbf{\Phi}_{mm}^{(k)} = \mathbf{M}_R^{(k)}\mathbf{\Phi}_{mm}^{(k)}\Lambda_{mm}^{(k)} \quad (14)$$

3. Present method

In this section, we present an improvement on previous substructuring techniques by modifying the iterative formula of the transformation matrices. The modification is based on Xia's method [12] for structural eigensolutions.

3.1 Governing equation for substructuring

We can rewrite Eq. (8) as

$$\begin{aligned}\mathbf{t}_{(1)} &= \mathbf{t}_G^{(1)} + \mathbf{t}_d^{(1)} \\ \mathbf{t}_{(2)} &= \mathbf{t}_G^{(2)} + \mathbf{t}_d^{(2)}\end{aligned} \quad (15a)$$

Where $\mathbf{T}_G^{(1)}$ and $\mathbf{T}_d^{(1)}$ represent the static and dynamic terms of the transformation matrix for substructure one as

$$\begin{aligned}\mathbf{t}_G^{(1)} &= -\left(\mathbf{K}_{ss}^{(1)}\right)^{-1}\mathbf{K}_{sm}^{(1)} \\ \mathbf{t}_d^{(1)} &= \left(\mathbf{K}_{ss}^{(1)}\right)^{-1}\left(\mathbf{M}_{sm}^{(1)} + \mathbf{M}_{ss}^{(1)}\mathbf{t}_{(1)}\right)\mathbf{\Phi}_{mm}\Lambda_{mm}\mathbf{\Phi}_{mm}^{-1}\end{aligned} \quad (15b)$$

And $\mathbf{T}_G^{(2)}$ and $\mathbf{T}_d^{(2)}$ also describe the static and dynamic terms of the transformation matrix for substructure two as

$$\begin{aligned}\mathbf{t}_G^{(2)} &= -\left(\mathbf{K}_{ss}^{(2)}\right)^{-1}\mathbf{K}_{sm}^{(2)} \\ \mathbf{t}_d^{(2)} &= \left(\mathbf{K}_{ss}^{(2)}\right)^{-1}\left(\mathbf{M}_{sm}^{(2)} + \mathbf{M}_{ss}^{(2)}\mathbf{t}_{(2)}\right)\mathbf{\Phi}_{mm}\Lambda_{mm}\mathbf{\Phi}_{mm}^{-1}\end{aligned} \quad (15c)$$

The above transformation matrices can be partitioned as

$$\begin{aligned}\mathbf{t}_{(1)} &= \begin{bmatrix}\mathbf{I}_{mm} \\ \mathbf{t}_{(1)}\end{bmatrix} = \begin{bmatrix}\mathbf{I}_{mm} \\ \mathbf{t}_G^{(1)} + \mathbf{t}_d^{(1)}\end{bmatrix} = \mathbf{t}_G^{(1)} + \begin{bmatrix}\mathbf{0} \\ \mathbf{t}_d^{(1)}\end{bmatrix} \\ \mathbf{t}_{(2)} &= \begin{bmatrix}\mathbf{I}_{mm} \\ \mathbf{t}_{(2)}\end{bmatrix} = \begin{bmatrix}\mathbf{I}_{mm} \\ \mathbf{t}_G^{(2)} + \mathbf{t}_d^{(2)}\end{bmatrix} = \mathbf{t}_G^{(2)} + \begin{bmatrix}\mathbf{0} \\ \mathbf{t}_d^{(2)}\end{bmatrix}\end{aligned} \quad (16)$$

Therefore, the global form of transformation matrix is given by

$$\mathbf{T} = \begin{bmatrix}\mathbf{t}_{(1)} \\ \mathbf{I}_{mm} \\ \mathbf{t}_{(2)}\end{bmatrix} = \begin{bmatrix}\mathbf{t}_G^{(1)} + \mathbf{t}_d^{(1)} \\ \mathbf{I}_{mm} \\ \mathbf{t}_G^{(2)} + \mathbf{t}_d^{(2)}\end{bmatrix} = \begin{bmatrix}\mathbf{t}_G^{(1)} \\ \mathbf{I}_{mm} \\ \mathbf{t}_G^{(2)}\end{bmatrix} + \begin{bmatrix}\mathbf{t}_d^{(1)} \\ \mathbf{0} \\ \mathbf{t}_d^{(2)}\end{bmatrix} \quad (17)$$

Substituting Eq. (17) into Eq. (10), one can get the reduced stiffness matrix as

$$\begin{aligned}\mathbf{K}_R &= \left(\begin{bmatrix}\mathbf{t}_G^{(1)} \\ \mathbf{I}_{mm} \\ \mathbf{t}_G^{(2)}\end{bmatrix} + \begin{bmatrix}\mathbf{t}_d^{(1)} \\ \mathbf{0} \\ \mathbf{t}_d^{(2)}\end{bmatrix}\right)^{\mathrm{T}} \mathbf{K} \left(\begin{bmatrix}\mathbf{t}_G^{(1)} \\ \mathbf{I}_{mm} \\ \mathbf{t}_G^{(2)}\end{bmatrix} + \begin{bmatrix}\mathbf{t}_d^{(1)} \\ \mathbf{0} \\ \mathbf{t}_d^{(2)}\end{bmatrix}\right) \\ &= \begin{bmatrix}\left(\mathbf{t}_G^{(1)}\right)^{\mathrm{T}} & \mathbf{I}_{mm} & \left(\mathbf{t}_G^{(2)}\right)^{\mathrm{T}}\end{bmatrix}\mathbf{K}\begin{bmatrix}\mathbf{t}_G^{(1)} \\ \mathbf{I}_{mm} \\ \mathbf{t}_G^{(2)}\end{bmatrix} \\ &\quad + \begin{bmatrix}\left(\mathbf{t}_G^{(1)}\right)^{\mathrm{T}} & \mathbf{I}_{mm} & \left(\mathbf{t}_G^{(2)}\right)^{\mathrm{T}}\end{bmatrix}\mathbf{K}\begin{bmatrix}\mathbf{t}_d^{(1)} \\ \mathbf{0} \\ \mathbf{t}_d^{(2)}\end{bmatrix} \\ &\quad + \begin{bmatrix}\left(\mathbf{t}_d^{(1)}\right)^{\mathrm{T}} & \mathbf{0} & \left(\mathbf{t}_d^{(2)}\right)^{\mathrm{T}}\end{bmatrix}\mathbf{K}\begin{bmatrix}\mathbf{t}_G^{(1)} \\ \mathbf{I}_{mm} \\ \mathbf{t}_G^{(2)}\end{bmatrix} \\ &\quad + \begin{bmatrix}\left(\mathbf{t}_d^{(1)}\right)^{\mathrm{T}} & \mathbf{0} & \left(\mathbf{t}_d^{(2)}\right)^{\mathrm{T}}\end{bmatrix}\mathbf{K}\begin{bmatrix}\mathbf{t}_d^{(1)} \\ \mathbf{0} \\ \mathbf{t}_d^{(2)}\end{bmatrix}\end{aligned}$$

$$= \mathbf{K}_G^{(1)} + \mathbf{K}_G^{(2)} + \begin{bmatrix} \left(\mathbf{t}_G^{(1)}\right)^{\mathrm{T}} & \mathbf{I}_{mm} & \left(\mathbf{t}_G^{(2)}\right)^{\mathrm{T}} \end{bmatrix} \tag{18}$$

$$\begin{bmatrix} \mathbf{K}_{ss}^{(1)} & \mathbf{K}_{sm}^{(1)} & \\ \mathbf{K}_{ms}^{(1)} & \mathbf{K}_{mm} & \mathbf{K}_{ms}^{(2)} \\ & \mathbf{K}_{sm}^{(2)} & \mathbf{K}_{ss}^{(2)} \end{bmatrix} \begin{bmatrix} \mathbf{t}_d^{(1)} \\ \mathbf{0} \\ \mathbf{t}_d^{(2)} \end{bmatrix} + \begin{bmatrix} \left(\mathbf{t}_d^{(1)}\right)^{\mathrm{T}} & \mathbf{0} & \left(\mathbf{t}_d^{(2)}\right)^{\mathrm{T}} \end{bmatrix}$$

$$\begin{bmatrix} \mathbf{K}_{ss}^{(1)} & \mathbf{K}_{sm}^{(1)} & \\ \mathbf{K}_{ms}^{(1)} & \mathbf{K}_{mm} & \mathbf{K}_{ms}^{(2)} \\ & \mathbf{K}_{sm}^{(2)} & \mathbf{K}_{ss}^{(2)} \end{bmatrix} \begin{bmatrix} \mathbf{t}_G^{(1)} \\ \mathbf{I}_{mm} \\ \mathbf{t}_G^{(2)} \end{bmatrix} + \begin{bmatrix} \left(\mathbf{t}_d^{(1)}\right)^{\mathrm{T}} & \mathbf{0} & \left(\mathbf{t}_d^{(2)}\right)^{\mathrm{T}} \end{bmatrix}$$

$$\begin{bmatrix} \mathbf{K}_{ss}^{(1)} & \mathbf{K}_{sm}^{(1)} & \\ \mathbf{K}_{ms}^{(1)} & \mathbf{K}_{mm} & \mathbf{K}_{ms}^{(2)} \\ & \mathbf{K}_{sm}^{(2)} & \mathbf{K}_{ss}^{(2)} \end{bmatrix} \begin{bmatrix} \mathbf{t}_d^{(1)} \\ \mathbf{0} \\ \mathbf{t}_d^{(2)} \end{bmatrix}$$

$$= \mathbf{K}_G^{(1)} + \mathbf{K}_G^{(2)} + \left\{ \left(\mathbf{t}_G^{(1)}\right)^{\mathrm{T}} \mathbf{K}_{ss}^{(1)} + \mathbf{K}_{ms}^{(1)} \right\} \mathbf{t}_d^{(1)} + \left\{ \mathbf{K}_{ms}^{(2)} + \left(\mathbf{t}_G^{(2)}\right)^{\mathrm{T}} \mathbf{K}_{ss}^{(2)} \right\} \mathbf{t}_d^{(2)} + \left(\mathbf{t}_d^{(1)}\right)^{\mathrm{T}} \left(\mathbf{K}_{ss}^{(1)} \mathbf{t}_G^{(1)} + \mathbf{K}_{sm}^{(1)} \right) + \left(\mathbf{t}_d^{(2)}\right)^{\mathrm{T}} \left(\mathbf{K}_{sm}^{(2)} + \mathbf{K}_{ss}^{(2)} \mathbf{t}_G^{(2)} \right) + \left(\mathbf{t}_d^{(1)}\right)^{\mathrm{T}} \mathbf{K}_{ss}^{(1)} \mathbf{t}_d^{(1)} + \left(\mathbf{t}_d^{(2)}\right)^{\mathrm{T}} \mathbf{K}_{ss}^{(2)} \mathbf{t}_d^{(2)}$$

Since $\left(\mathbf{t}_G^{(1)}\right)^{\mathrm{T}} \mathbf{K}_{ss}^{(1)} + \mathbf{K}_{ms}^{(1)} = \mathbf{K}_{ms}^{(2)} + \left(\mathbf{t}_G^{(2)}\right)^{\mathrm{T}} \mathbf{K}_{ss}^{(2)} = 0$ and $\mathbf{K}_{ss}^{(1)} \mathbf{t}_G^{(1)} + \mathbf{K}_{sm}^{(1)} = \mathbf{K}_{sm}^{(2)} + \mathbf{K}_{ss}^{(2)} \mathbf{t}_G^{(2)} = 0$, the above equation becomes

$$\mathbf{K}_R = \mathbf{K}_G^{(1)} + \mathbf{K}_G^{(2)} + \left(\mathbf{t}_d^{(1)}\right)^{\mathrm{T}} \mathbf{K}_{ss}^{(1)} \mathbf{t}_d^{(1)} + \left(\mathbf{t}_d^{(2)}\right)^{\mathrm{T}} \mathbf{K}_{ss}^{(2)} \mathbf{t}_d^{(2)} \tag{19}$$

Similarly, we can obtain the reduced mass matrix as

$$\mathbf{M}_R = \mathbf{M}_G^{(1)} + \mathbf{M}_G^{(2)} + \left\{ \left(\mathbf{t}_G^{(1)}\right)^{\mathrm{T}} \mathbf{M}_{ss}^{(1)} + \mathbf{M}_{ms}^{(1)} \right\} \mathbf{t}_d^{(1)} + \left\{ \mathbf{M}_{ms}^{(2)} + \left(\mathbf{t}_G^{(2)}\right)^{\mathrm{T}} \mathbf{M}_{ss}^{(2)} \right\} \mathbf{t}_d^{(2)} + \left(\mathbf{t}_d^{(1)}\right)^{\mathrm{T}} \left(\mathbf{M}_{ss}^{(1)} \mathbf{t}_G^{(1)} + \mathbf{M}_{sm}^{(1)} \right) + \left(\mathbf{t}_d^{(2)}\right)^{\mathrm{T}} \left(\mathbf{M}_{sm}^{(2)} + \mathbf{M}_{ss}^{(2)} \mathbf{t}_G^{(2)} \right) + \left(\mathbf{t}_d^{(1)}\right)^{\mathrm{T}} \mathbf{M}_{ss}^{(1)} \mathbf{t}_d^{(1)} + \left(\mathbf{t}_d^{(2)}\right)^{\mathrm{T}} \mathbf{M}_{ss}^{(2)} \mathbf{t}_d^{(2)} \tag{20}$$

Therefore, Eq. (11) can be rewritten as

$$\left\{ \mathbf{K}_G^{(1)} + \mathbf{K}_G^{(2)} + \left(\mathbf{t}_d^{(1)}\right)^{\mathrm{T}} \mathbf{K}_{ss}^{(1)} \mathbf{t}_d^{(1)} + \left(\mathbf{t}_d^{(2)}\right)^{\mathrm{T}} \mathbf{K}_{ss}^{(2)} \mathbf{t}_d^{(2)} \right\} \mathbf{\Phi}_{mm} - \Big[\mathbf{M}_G^{(1)} + \mathbf{M}_G^{(2)} + \left\{ \left(\mathbf{t}_G^{(1)}\right)^{\mathrm{T}} \mathbf{M}_{ss}^{(1)} + \mathbf{M}_{ms}^{(1)} \right\} \mathbf{t}_d^{(1)} + \left\{ \mathbf{M}_{ms}^{(2)} + \left(\mathbf{t}_G^{(2)}\right)^{\mathrm{T}} \mathbf{M}_{ss}^{(2)} \right\} \mathbf{t}_d^{(2)} + \left(\mathbf{t}_d^{(1)}\right)^{\mathrm{T}} \left(\mathbf{M}_{ss}^{(1)} \mathbf{t}_G^{(1)} + \mathbf{M}_{sm}^{(1)} \right) + \left(\mathbf{t}_d^{(2)}\right)^{\mathrm{T}} \left(\mathbf{M}_{sm}^{(2)} + \mathbf{M}_{ss}^{(2)} \mathbf{t}_G^{(2)} \right) + \left(\mathbf{t}_d^{(1)}\right)^{\mathrm{T}} \mathbf{M}_{ss}^{(1)} \mathbf{t}_d^{(1)} + \left(\mathbf{t}_d^{(2)}\right)^{\mathrm{T}} \mathbf{M}_{ss}^{(2)} \mathbf{t}_d^{(2)} \Big] \mathbf{\Phi}_{mm} \Lambda_{mm} = 0 \tag{21}$$

On the first term of the above equation $\mathbf{T}_d^{(1)}$ and $\mathbf{T}_d^{(2)}$ are substituted by Eq. (15b) and Eq. (15c),

$$\begin{aligned} &\left\{ \left(\mathbf{t}_d^{(1)}\right)^{\mathrm{T}} \mathbf{K}_{ss}^{(1)} \mathbf{t}_d^{(1)} + \left(\mathbf{t}_d^{(2)}\right)^{\mathrm{T}} \mathbf{K}_{ss}^{(2)} \mathbf{t}_d^{(2)} \right\} \mathbf{\Phi}_{mm} \\ &= \Big[\left(\mathbf{t}_d^{(1)}\right)^{\mathrm{T}} \mathbf{K}_{ss}^{(1)} \Big\{ \left(\mathbf{K}_{ss}^{(1)}\right)^{-1} \left(\mathbf{M}_{sm}^{(1)} + \mathbf{M}_{ss}^{(1)} \mathbf{t}_G^{(1)} \right) + \left(\mathbf{K}_{ss}^{(1)}\right)^{-1} \mathbf{M}_{ss}^{(1)} \mathbf{t}_d^{(1)} \Big\} \mathbf{\Phi}_{mm} \Lambda_{mm} \mathbf{\Phi}_{mm}^{-1} \\ &+ \left(\mathbf{t}_d^{(2)}\right)^{\mathrm{T}} \mathbf{K}_{ss}^{(2)} \Big\{ \left(\mathbf{K}_{ss}^{(2)}\right)^{-1} \left(\mathbf{M}_{sm}^{(2)} + \mathbf{M}_{ss}^{(2)} \mathbf{t}_G^{(2)} \right) + \left(\mathbf{K}_{ss}^{(2)}\right)^{-1} \mathbf{M}_{ss}^{(2)} \mathbf{t}_d^{(2)} \Big\} \mathbf{\Phi}_{mm} \Lambda_{mm} \mathbf{\Phi}_{mm}^{-1} \Big] \mathbf{\Phi}_{mm} \\ &= \left(\mathbf{t}_d^{(1)}\right)^{\mathrm{T}} \left(\mathbf{M}_{sm}^{(1)} + \mathbf{M}_{ss}^{(1)} \mathbf{t}_G^{(1)} \right) \mathbf{\Phi}_{mm} \Lambda_{mm} + \left(\mathbf{t}_d^{(1)}\right)^{\mathrm{T}} \mathbf{M}_{ss}^{(1)} \mathbf{t}_d^{(1)} \mathbf{\Phi}_{mm} \Lambda_{mm} \\ &+ \left(\mathbf{t}_d^{(2)}\right)^{\mathrm{T}} \left(\mathbf{M}_{sm}^{(2)} + \mathbf{M}_{ss}^{(2)} \mathbf{t}_G^{(2)} \right) \mathbf{\Phi}_{mm} \Lambda_{mm} + \left(\mathbf{t}_d^{(2)}\right)^{\mathrm{T}} \mathbf{M}_{ss}^{(2)} \mathbf{t}_d^{(2)} \mathbf{\Phi}_{mm} \Lambda_{mm} \end{aligned} \tag{22}$$

Substituting Eq. (A15) into Eq. (A14) and removing the identical terms, we can get

$$\begin{aligned} &\mathbf{K}_G^{(1)} \mathbf{\Phi}_{mm} + \mathbf{K}_G^{(2)} \mathbf{\Phi}_{mm} - \Big[\mathbf{M}_G^{(1)} + \left\{ \mathbf{M}_{ms}^{(1)} + \left(\mathbf{t}_G^{(1)}\right)^{\mathrm{T}} \mathbf{M}_{ss}^{(1)} \right\} \mathbf{t}_d^{(1)} + \mathbf{M}_G^{(2)} + \left\{ \mathbf{M}_{ms}^{(2)} + \left(\mathbf{t}_G^{(2)}\right)^{\mathrm{T}} \mathbf{M}_{ss}^{(2)} \right\} \mathbf{t}_d^{(2)} \Big] \mathbf{\Phi}_{mm} \Lambda_{mm} \\ &= \left(\mathbf{K}_G^{(1)} \mathbf{\Phi}_{mm} - \mathbf{M}_d^{(1)} \mathbf{\Phi}_{mm} \Lambda_{mm} \right) + \left(\mathbf{K}_G^{(2)} \mathbf{\Phi}_{mm} - \mathbf{M}_d^{(2)} \mathbf{\Phi}_{mm} \Lambda_{mm} \right) \\ &= \left(\mathbf{K}_G^{(1)} + \mathbf{K}_G^{(2)} \right) \mathbf{\Phi}_{mm} - \left(\mathbf{M}_d^{(1)} + \mathbf{M}_d^{(2)} \right) \mathbf{\Phi}_{mm} \Lambda_{mm} \\ &= \mathbf{K}_G \mathbf{\Phi}_{mm} - \mathbf{M}_d \mathbf{\Phi}_{mm} \Lambda_{mm} = 0 \end{aligned} \tag{23}$$

Where

$$
\begin{aligned}
\mathbf{M}_d &= \mathbf{M}_d^{(1)} + \mathbf{M}_d^{(2)} \\
&= \mathbf{M}_G^{(1)} + \left\{ \mathbf{M}_{ms}^{(1)} + \left(\mathbf{t}_G^{(1)}\right)^{\mathrm{T}} \mathbf{M}_{ss}^{(1)} \right\} \mathbf{t}_d^{(1)} + \mathbf{M}_G^{(2)} \\
&\quad + \left\{ \mathbf{M}_{ms}^{(2)} + \left(\mathbf{t}_G^{(2)}\right)^{\mathrm{T}} \mathbf{M}_{ss}^{(2)} \right\} \mathbf{t}_d^{(2)} \\
&= \begin{bmatrix} \mathbf{t}_G^{(1)} & \mathbf{I}_{mm} & \mathbf{t}_G^{(2)} \end{bmatrix} \begin{bmatrix} \mathbf{M}_{ss}^{(1)} & \mathbf{M}_{sm}^{(1)} & \\ \mathbf{M}_{ms}^{(1)} & \mathbf{M}_{mm} & \mathbf{M}_{ms}^{(2)} \\ & \mathbf{M}_{sm}^{(2)} & \mathbf{M}_{ss}^{(2)} \end{bmatrix} \\
&\quad \begin{bmatrix} \mathbf{t}_G^{(1)} \\ \mathbf{I}_{mm} \\ \mathbf{t}_G^{(2)} \end{bmatrix} + \begin{bmatrix} \mathbf{t}_G^{(1)} & \mathbf{I}_{mm} & \mathbf{t}_G^{(2)} \end{bmatrix} \\
&\quad \begin{bmatrix} \mathbf{M}_{ss}^{(1)} & \mathbf{M}_{sm}^{(1)} & \\ \mathbf{M}_{ms}^{(1)} & \mathbf{M}_{mm} & \mathbf{M}_{ms}^{(2)} \\ & \mathbf{M}_{sm}^{(2)} & \mathbf{M}_{ss}^{(2)} \end{bmatrix} \begin{bmatrix} \mathbf{t}_d^{(1)} \\ \mathbf{0} \\ \mathbf{t}_d^{(2)} \end{bmatrix}
\end{aligned}
\tag{24}
$$

Therefore, from Eq. (23), we can get

$$\mathbf{\Phi}_{mm} \Lambda_{mm} \mathbf{\Phi}_{mm}^{-1} = \mathbf{M}_d^{-1} \mathbf{K}_G \tag{25}$$

Substituting Eq. (25) into Eq. (15b) and Eq. (15c) then from Eq. (15a)

$$
\begin{aligned}
\mathbf{t}_{(1)} &= \mathbf{t}_G^{(1)} + \mathbf{t}_d^{(1)} = \mathbf{t}_G^{(1)} + \left(\mathbf{K}_{ss}^{(1)}\right)^{-1} \left(\mathbf{M}_{sm}^{(1)} + \mathbf{M}_{ss}^{(1)} \mathbf{t}_{(1)}\right) \mathbf{M}_d^{-1} \mathbf{K}_G \\
\mathbf{t}_{(2)} &= \mathbf{t}_G^{(2)} + \mathbf{t}_d^{(2)} = \mathbf{t}_G^{(2)} + \left(\mathbf{K}_{ss}^{(2)}\right)^{-1} \left(\mathbf{M}_{sm}^{(2)} + \mathbf{M}_{ss}^{(2)} \mathbf{t}_{(2)}\right) \mathbf{M}_d^{-1} \mathbf{K}_G
\end{aligned}
\tag{26}
$$

The detailed iterative procedure is shown in the Appendix. From Eq. (A9 ~ A10), the lowest m eigenvalues and the associated eigenvector after (k-1)th iteration are estimated by solving the generalized eigenproblem of the reduced system:

$$\mathbf{K}_R^{(k)} \mathbf{\Phi}_{mm}^{(k)} = \mathbf{M}_R^{(k)} \mathbf{\Phi}_{mm}^{(k)} \Lambda_{mm}^{(k)} \tag{27}$$

3.2 Iterative scheme for the present method

The main steps for improved iterative substructuring are as follows:

Step 1: Separate the whole structure into two (or more) smaller substructures.

Step 2: Choose the master degrees of freedom including interface degrees of freedom in each substructure and compute all the submatrices to be used in the following.

Step 3: Calculate the initial approximation of the transformation matrices $\mathbf{t}_{(1)}^{(0)}, \mathbf{t}_{(2)}^{(0)}, \mathbf{t}_{(3)}^{(0)} \cdots, \mathbf{t}_G^{(1)}, \mathbf{t}_G^{(2)}, \mathbf{t}_G^{(3)} \cdots$ in each substructure by using Eq. (A12).

Step 4: Construct the Guyan reduction stiffness matrices in each substructure and assemble them into one, $\mathbf{K}_{Guyan}$, i.e. by using Eq. (A13).

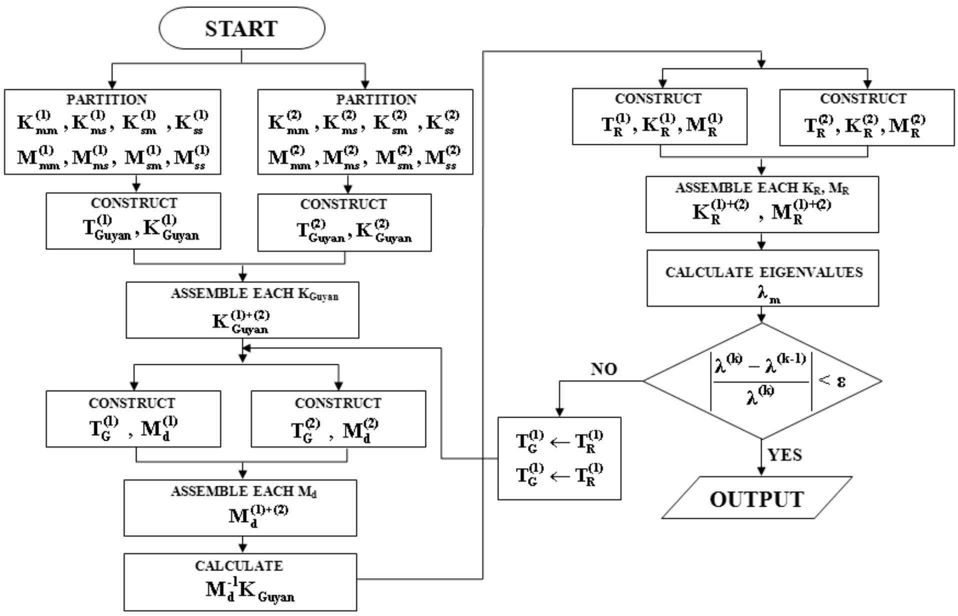


Fig. 1. Flowchart of the present substructuring reduction procedure.

Step 5: Calculate the initial approximation of $\mathbf{M}_d^{(0)}$ in each substructure and assemble them into one by using Eq. (A14). The initial mass matrix $\mathbf{M}_d^{(0)}$ becomes Guyan reduction mass matrix, i.e., $\mathbf{M}_{Guyan}$.

Step 6: Calculate the initial transformation matrices $\mathbf{t}_{(1)}^{(1)}, \mathbf{t}_{(2)}^{(1)}, \mathbf{t}_{(3)}^{(1)} \cdots$ in each substructure by using Eq. (A15).

Step 7: Construct the system matrices of reduced model in each substructure and assemble them into one using Eq. (A16).

Step 8: Solve for eigenproblem of the reduced model using Eq. (27).

Step 9: Check the convergence for the eigenvalue by using the following convergent criterion:

$$\frac{\left|\lambda_j^{(k)} - \lambda_j^{(k-1)}\right|}{\left|\lambda_j^{(k)}\right|} \le \varepsilon \quad , \quad j = 1, 2, ..., p \tag{28}$$

where λ denotes the eigenvalues. If the p eigenvalues converge, exit the loop. If not, go back to Step 5.

Step 10: Output the system matrices $\mathbf{K}_R^{(k)}$ and $\mathbf{M}_R^{(k)}$ of the reduced model.

Fig. 1 shows the main procedure for the present method. The kernel of it is the repeated update of $\mathbf{M}_d$ matrix in each substructure. And the major difference of the present algorithms from the previous method is that the reduced matrices $\mathbf{K}_R$, $\mathbf{M}_R$ are not used in the next iteration step but only transformation matrices and Guyan reduced stiffness matrix are required during iteration.

4. Numerical examples

To illustrate the convergence and effectiveness of the proposed method, two numerical examples are considered. In these examples, the "convergence" implies that the eigenvalues and eigenvectors calculated from the reduced system are approaching those obtained from the global system. Therefore, the absolute values of relative errors are defined as

$$relative\ error : \varepsilon_\omega = \frac{\left|\omega_{reduced} - \omega_{full}\right|}{\omega_{full}} \times 100 \tag{29}$$

In Eq. (29), ω_{full} and $\omega_{reduced}$ are the modal frequencies calculated from the global system and the reduced system, respectively.

4.1 Isotropic cantilever beam structure

An isotropic cantilever beam structure using quadrilateral elements is shown in Fig. 2 (a). The cantile-

Table 1. Comparison of the number of d.o.f. in full system and in subsystem and the size of the transformation matrix in the cantilever beam structure.

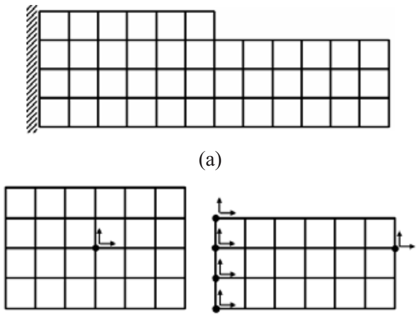
		Total d.o.f.	Master d.o.f.	Slave d.o.f.	Interface d.o.f.	Transformation matrix
Full system		118	12	106	0	[106×12]
Subsystem	Sub-1	70	4	60	8	[70×12]
	Sub-2	56	4	46	8	[56×12]

Table 2. First ten modal frequencies (rad/s) of the cantilever beam structure obtained with the previous method.

Iter-ation	Mode									
	1	2	3	4	5	6	7	8	9	10
0	3.4918	13.6371	16.8815	31.0197	45.8617	51.5021	75.9484	81.3397	113.5585	116.5458
1			16.8809	30.9531	45.4761	49.7445	74.2669	78.3685	93.5442	106.3656
2			16.8808	30.9449	45.3799	49.4275	73.8658	76.7021	86.7236	101.9934
3				30.9410	45.3384	49.2857	73.6930	75.4345	83.9084	98.7716
4				30.9385	45.3147	49.2000	73.5791	74.5751	82.4995	96.3340
5				30.9368	45.2993	49.1418	73.4647	74.0545	81.6902	94.5109
6				30.9355	45.2885	49.0995	73.3147	73.7935	81.1905	93.1880
7				30.9345	45.2805	49.0674	73.1538	73.6809	80.8636	92.2372
8				30.9337	45.2742	49.0420	73.0176	73.6259	80.6379	91.5509
9				30.9330	45.2693	49.0215	72.9094	73.5931	80.4738	91.0506
10				30.9325	45.2652	49.0045	72.8226	73.5708	80.3490	90.6813
exact	3.4918	13.6371	16.8808	30.9275	45.2260	48.8442	71.9178	73.4500	79.2529	88.8576

Table 3. First ten modal frequencies (rad/s) of the cantilever beam structure obtained with the present method.

Iteration	Mode									
	1	2	3	4	5	6	7	8	9	10
0	3.4918	13.6371	16.8815	31.0197	45.8617	51.5021	75.9484	81.3397	113.5585	116.5458
1			16.8808	30.9291	45.2934	49.0204	73.5095	77.0512	87.1083	102.8554
2				30.9275	45.2316	48.8626	73.1776	74.8780	82.2217	98.3125
3					45.2264	48.8459	72.8061	73.7684	80.6829	95.4252
4					45.2261	48.8443	72.3186	73.5270	79.9710	93.2519
5					45.2260	48.8442	72.0614	73.4738	79.5989	91.6463
6							71.9638	73.4578	79.4117	90.9534
7							71.9317	73.4525	79.3231	89.9367
8							71.9218	73.4508	79.2832	89.5327
9							71.9189	73.4503	79.2658	89.2837
10							71.9181	73.4501	79.2583	89.1290
Exact	3.4918	13.6371	16.8808	30.9275	45.2260	48.8442	71.9178	73.4500	79.2529	88.8576



(a)

(b)

Fig. 2. Finite element model of the cantilever beam structure and selection of master d.o.f. in each substructure (E=4MPa, ρ=2800 kg/m^3, ν=0.3).

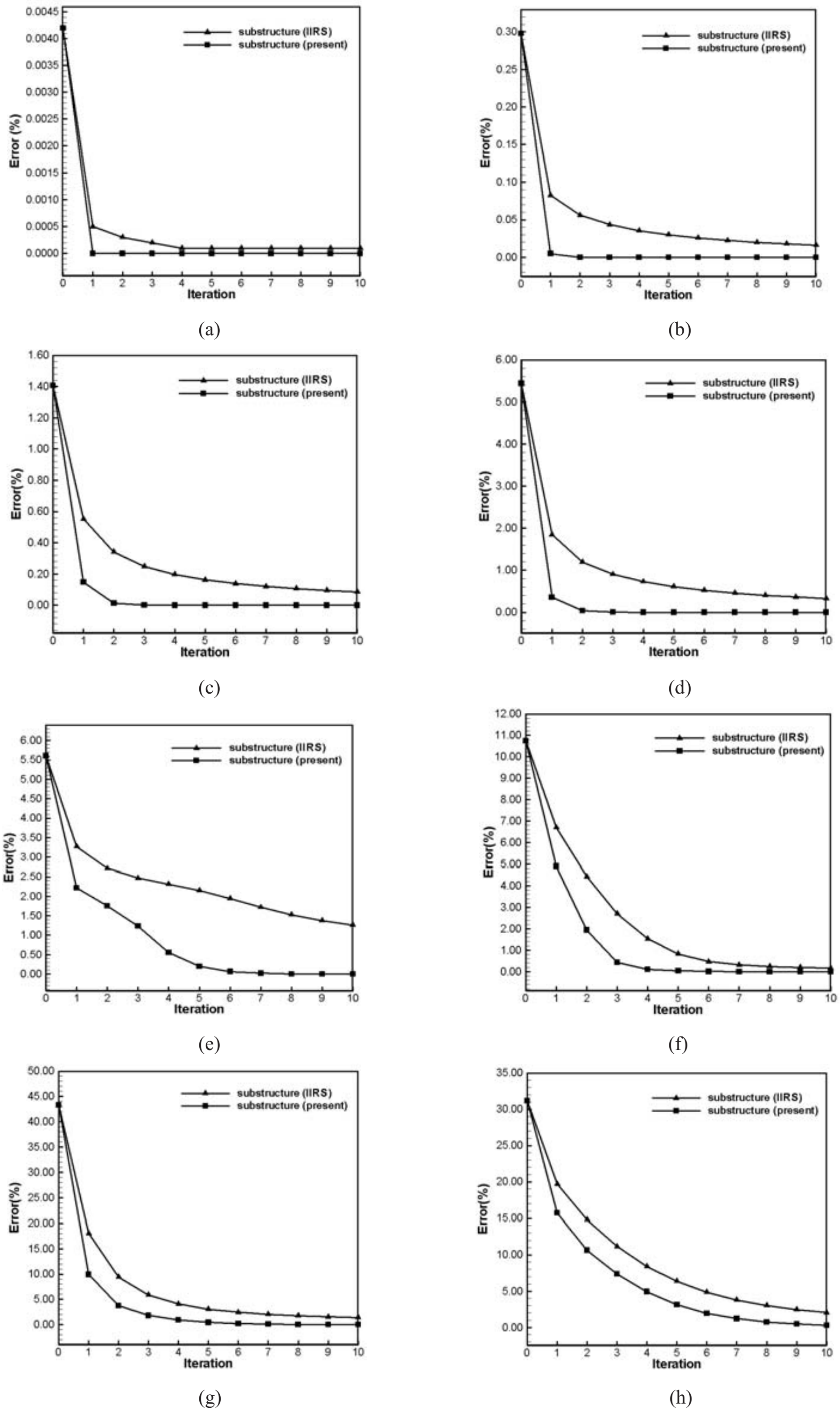


Fig. 3. Percent errors of the third through tenth modal frequencies of the cantilever beam structure: (a) third modal frequency; (b) fourth modal frequency; (c) fifth modal frequency; (d) sixth modal frequency; (e) seventh modal frequency; (f) eighth modal frequency; (g) ninth modal frequency; (h) tenth modal frequency.

ver beam is constrained at the left end side, and it contains a total of 59 nodes, 42 elements and 118 degrees of freedom. To apply the substructuring technique to the beam structure, the whole system is divided into two substructures. As shown in Fig. 2 (b), a total of 12 arbitrary degrees of freedom at the six specified nodes is selected as the master degrees of freedom including the interface degrees of freedom. The final reduced system is 10.2% of the global system. Table 1 shows the number of degrees of freedom and the size of transformation matrices in full system and in each subsystem. It can be clearly seen that the size of the full system is reduced to the size of each subsystem. The first ten modal frequencies of two of the reduced models of the previous and the present method are listed in Table 2 and Table 3. The modal frequencies of both tables converge to the global ones as the iterations continue. But the results of Table 3 are much faster in convergence speed. Fig. 3 shows a comparison of the percent errors of the third through tenth modal frequencies. From this figure, the present method can save lots of computational cost.

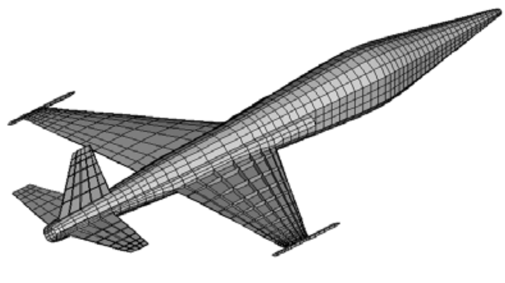
4.2 Jet fighter structure

A conventional jet fighter structure clamped along nozzle section, shown in Fig. 4 (a), is considered. Aminpour's shell plate with 6 degrees of freedom per node [22] is used. The model contains a total of 2,536 nodes, 3,158 elements and 15,216 degrees of freedom. And the structure is divided into five substructures. Fig. 4 (b) shows the result of the selection of the master degrees of freedom including interface degrees of freedom. A total of 570 degrees of freedom is selected as the master degree of freedom out of each substructure with the arbitrary selection. This problem is condensed to the reduced one with 3.7% degrees of freedom of the full system. Table 4 represents the number of degrees of freedom and the size of transformation matrices in full system and in each subsystem.

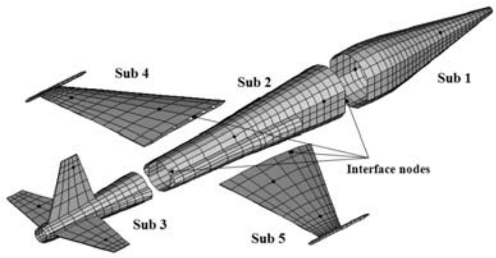
The results in Table 5 show the first thirty modal frequencies by using the previous substructuring technique. All the modal frequencies are converged to the global ones when the iterations are continued. Table 6 shows the first thirty modal frequencies predicted by the present method. As shown clearly from the result in Table 6, more accurate modal frequencies are calculated from just the first iteration in comparison with the results of Table 5. In Table 7, the computation times of the previous method and the present substructuring scheme are compared. Although the present procedure is a little complicated and it requires more computation time for iteration, the speed of convergence is very fast. Thus, it takes less time as a whole to construct a highly accurate reduced system. Fig. 5 shows the relative errors of modal frequencies of the two substructuring methods. From this figure, the previous method may need more iteration to obtain the accuracy level of the present scheme. Consequently, the present method is more efficient in the iterative approach for dynamic condensation.

Table 4. First thirty modal frequencies (rad/s) of the jet fighter structure obtained with the previous method.

		Total d.o.f.	Master d.o.f.	Slave d.o.f.	Interface d.o.f.	Transformation matrix
Full system		15,216	570	14,646	0	[14,646×570]
Sub-system	Sub-1	3,720	12	3,588	120	[3,720×570]
	Sub-2	3,528	12	3,012	504	[3,528×570]
	Sub-3	3,840	12	3,638	192	[3,840×570]
	Sub-4	2,328	12	2,124	192	[2,328×570]
	Sub-5	2,328	12	2,124	192	[2,328×570]



(a)



(b)

Fig. 4. Finite element model of the jet fighter structure and selection of master d.o.f. in each substructure (E=72 GPa, ρ=2800 kg/m^3, ν=0.3).

Table 5. First thirty modal frequencies (rad/s) of the jet fighter structure obtained with the previous method.

Iteration	Mode									
	1	2	3	4	5	6	7	8	9	10
1	12.4225	15.0360	25.2254	37.3643	70.9719	71.9537	77.9250	85.8091	90.7740	99.6462
2					70.9719	71.9537	77.9249	85.8088	90.7733	99.6434
3					70.9718	71.9536	77.9249	85.8086	90.7729	99.6421
exact	12.4225	15.0360	25.2254	37.3643	70.9719	71.9535	77.9248	85.8078	90.7714	99.6376
Iteration	Mode									
	11	12	13	14	15	16	17	18	19	20
1	101.6078	115.2883	119.3945	121.4479	129.1902	129.9551	130.4707	143.6789	146.1033	153.3050
2	101.6029	115.2858	119.3655	121.4016	129.1348	129.8976	130.4129	143.4128	145.3671	152.4147
3	101.6008	115.2845	119.3538	121.3840	129.1111	129.8679	130.3899	143.2771	144.9871	152.1501
exact	101.5932	115.2815	119.2917	121.2935	129.0730	129.8005	130.3323	142.5237	142.5271	150.6606
Iteration	Mode									
	21	22	23	24	25	26	27	28	29	30
1	154.8380	165.7972	167.4776	169.6785	193.3589	197.1594	201.1999	206.5867	207.0341	209.5591
2	153.9828	165.7063	167.1898	169.2909	192.9241	197.1503	200.8712	205.7874	206.4909	208.1671
3	153.7165	165.6690	167.0599	169.1031	192.7385	197.1464	200.7151	205.5122	206.2693	207.8510
exact	152.2825	165.6038	166.3418	168.0545	191.5554	197.1244	200.2674	203.9085	205.2761	206.7073

Table 6. First thirty modal frequencies (rad/s) of the jet fighter structure obtained with the present method.

Iteration	Mode									
	1	2	3	4	5	6	7	8	9	10
1	12.4225	15.0360	25.2254	37.3643	70.9718	71.9536	77.9248	85.8079	90.7715	99.6379
exact	12.4225	15.0360	25.2254	37.3643	70.9719	71.9535	77.9248	85.8078	90.7714	99.6376
Iteration	Mode									
	11	12	13	14	15	16	17	18	19	20
1	101.5942	115.2817	119.3040	121.3137	129.0809	129.8075	130.3437	142.5459	142.6786	151.1145
exact	101.5932	115.2815	119.2917	121.2935	129.0730	129.8005	130.3323	142.5237	142.5271	150.6606
Iteration	Mode									
	21	22	23	24	25	26	27	28	29	30
1	152.8595	165.6207	166.4689	168.2076	192.5387	197.1282	200.3928	204.4102	205.9728	207.4829
exact	152.2825	165.6038	166.3418	168.0545	191.5554	197.1244	200.2674	203.9085	205.2761	206.7073

Table 7. Time comparison between the previous method and the present method.

Substructuring Technique	Previous Method (IIRS)	Present Method (Accelerated IIRS)
Time Cost (sec)	1664	938
Max. Freq. Error (%)	1.73	0.51

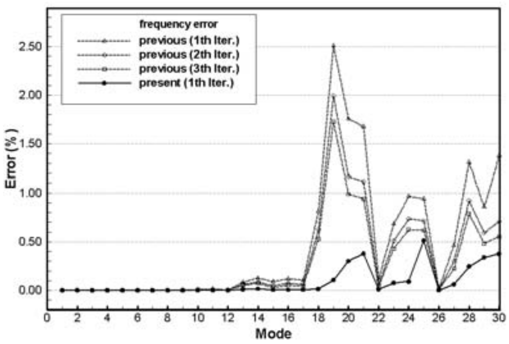


Fig. 5. Comparison of relative errors of the jet fighter structure with the previous and the present method.

5. Conclusion

A new accelerated iterative substructuring reduction technique for dynamic condensation is developed. This method improves the previous substructuring scheme combined with the iterated IRS method. The convergence and effectiveness of this method are verified through the two numerical examples. As compared with the previous iterative substructuring method, the proposed method converges much faster to the global values and saves considerable computational cost. The key point of the present method is in iteratively updating the transformation matrix in each substructure. Since the present scheme does not require large number of iterations, the present method is effectively applicable to the dynamic analysis for large structures even under the environment of limited computer storage. Therefore, the present scheme is applicable to structural optimization, system identification and vibration analysis and control.

Acknowledgments

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Appendix

The iterative substructuring reduction formulation for the previous and the present methods is derived below.

A.1 Iterative substructuring procedure for the IIRS method

The iterative forms of Eq. (13) for k=1, 2, 3... , can be expressed by

$$\begin{aligned} \mathbf{t}_{(1)}^{(k)} &= -\left(\mathbf{K}_{ss}^{(1)}\right)^{-1}\mathbf{K}_{sm}^{(1)} + \left(\mathbf{K}_{ss}^{(1)}\right)^{-1} \\ &\quad \left(\mathbf{M}_{sm}^{(1)} + \mathbf{M}_{ss}^{(1)}\mathbf{t}_{(1)}^{(k-1)}\right)\left(\mathbf{M}_{R}^{(k-1)}\right)^{-1}\mathbf{K}_{R}^{(k-1)} \\ \mathbf{t}_{(2)}^{(k)} &= -\left(\mathbf{K}_{ss}^{(2)}\right)^{-1}\mathbf{K}_{sm}^{(2)} + \left(\mathbf{K}_{ss}^{(2)}\right)^{-1} \\ &\quad \left(\mathbf{M}_{sm}^{(2)} + \mathbf{M}_{ss}^{(2)}\mathbf{t}_{(2)}^{(k-1)}\right)\left(\mathbf{M}_{R}^{(k-1)}\right)^{-1}\mathbf{K}_{R}^{(k-1)} \end{aligned} \tag{A1}$$

where the superscript "k" represents the (k-1)th iteration. If the initial approximations of the transformation matrices are given by

$$\begin{aligned} \mathbf{t}_{(1)}^{(0)} &= -\left(\mathbf{K}_{ss}^{(1)}\right)^{-1}\mathbf{K}_{sm}^{(1)} \\ \mathbf{t}_{(2)}^{(0)} &= -\left(\mathbf{K}_{ss}^{(2)}\right)^{-1}\mathbf{K}_{sm}^{(2)} \end{aligned} \tag{A2}$$

These are the Guyan transformation matrices ignoring the dynamic part of Eq. (A1). By these transformation matrices, the Guyan reduction matrices are constructed as follows:

$$\begin{aligned} \mathbf{K}_{Guyan} &= \left[\left(\mathbf{t}_{(1)}^{(0)}\right)^{T} \quad \mathbf{I}_{mm} \quad \left(\mathbf{t}_{(2)}^{(0)}\right)^{T}\right]\begin{bmatrix} \mathbf{K}_{ss}^{(1)} & \mathbf{K}_{sm}^{(1)} & \\ \mathbf{K}_{ms}^{(1)} & \mathbf{K}_{mm} & \mathbf{K}_{ms}^{(2)} \\ & \mathbf{K}_{sm}^{(2)} & \mathbf{K}_{ss}^{(2)} \end{bmatrix} \\ &\quad \begin{bmatrix} \mathbf{t}_{(1)}^{(0)} \\ \mathbf{I}_{mm} \\ \mathbf{t}_{(2)}^{(0)} \end{bmatrix} = \left(\mathbf{t}_{(1)}^{(0)}\right)^{T}\mathbf{K}_{ss}^{(1)}\mathbf{t}_{(1)}^{(0)} + \mathbf{K}_{ms}^{(1)}\mathbf{t}_{(1)}^{(0)} + \left(\mathbf{t}_{(1)}^{(0)}\right)^{T}\mathbf{K}_{sm}^{(1)} \\ &\quad + \mathbf{K}_{mm} + \left(\mathbf{t}_{(2)}^{(0)}\right)^{T}\mathbf{K}_{sm}^{(2)} + \mathbf{K}_{ms}^{(2)}\mathbf{t}_{(2)}^{(0)} + \left(\mathbf{t}_{(2)}^{(0)}\right)^{T}\mathbf{K}_{ss}^{(2)}\mathbf{t}_{(2)}^{(0)} \\ \mathbf{M}_{Guyan} &= \left[\left(\mathbf{t}_{(1)}^{(0)}\right)^{T} \quad \mathbf{I}_{mm} \quad \left(\mathbf{t}_{(2)}^{(0)}\right)^{T}\right]\begin{bmatrix} \mathbf{M}_{ss}^{(1)} & \mathbf{M}_{sm}^{(1)} & \\ \mathbf{M}_{ms}^{(1)} & \mathbf{M}_{mm} & \mathbf{M}_{ms}^{(2)} \\ & \mathbf{M}_{sm}^{(2)} & \mathbf{M}_{ss}^{(2)} \end{bmatrix} \\ &\quad \begin{bmatrix} \mathbf{t}_{(1)}^{(0)} \\ \mathbf{I}_{mm} \\ \mathbf{t}_{(2)}^{(0)} \end{bmatrix} = \left(\mathbf{t}_{(1)}^{(0)}\right)^{T}\mathbf{M}_{ss}^{(1)}\mathbf{t}_{(1)}^{(0)} + \mathbf{M}_{ms}^{(1)}\mathbf{t}_{(1)}^{(0)} + \left(\mathbf{t}_{(1)}^{(0)}\right)^{T}\mathbf{M}_{sm}^{(1)} \\ &\quad + \mathbf{M}_{mm} + \left(\mathbf{t}_{(2)}^{(0)}\right)^{T}\mathbf{M}_{sm}^{(2)} + \mathbf{M}_{ms}^{(2)}\mathbf{t}_{(2)}^{(0)} + \left(\mathbf{t}_{(2)}^{(0)}\right)^{T}\mathbf{M}_{ss}^{(2)}\mathbf{t}_{(2)}^{(0)} \end{aligned} \tag{A3}$$

As shown Eq. (A3), the Guyan reduction matrices are constructed in the substructure level and these reduced matrices are assembled into one reduced system. For an iterative dynamic condensation, $\mathbf{K}_{\text{Guyan}}$ and $\mathbf{M}_{\text{Guyan}}$ become the starting reduced system matrices as

$$\begin{aligned} \mathbf{K}_{R}^{(0)} &= \mathbf{K}_{Guyan} \\ \mathbf{M}_{R}^{(0)} &= \mathbf{M}_{Guyan} \end{aligned} \tag{A4}$$

Substituting Eq. (A4) into Eq. (A1) and using Eq. (A2), the initial transformation matrices for k=1, i.e. 0th iteration can be constructed as

$$\begin{aligned} \mathbf{t}_{(1)}^{(1)} &= -\left(\mathbf{K}_{ss}^{(1)}\right)^{-1}\mathbf{K}_{sm}^{(1)} + \left(\mathbf{K}_{ss}^{(1)}\right)^{-1}\left(\mathbf{M}_{ss}^{(1)}\mathbf{t}_{(1)}^{(0)} + \mathbf{M}_{sm}^{(1)}\right) \\ &\quad \left(\mathbf{M}_{R}^{(0)}\right)^{-1}\mathbf{K}_{R}^{(0)} \\ \mathbf{t}_{(2)}^{(1)} &= -\left(\mathbf{K}_{ss}^{(2)}\right)^{-1}\mathbf{K}_{sm}^{(2)} + \left(\mathbf{K}_{ss}^{(2)}\right)^{-1}\left(\mathbf{M}_{ss}^{(2)}\mathbf{t}_{(2)}^{(0)} + \mathbf{M}_{sm}^{(2)}\right) \\ &\quad \left(\mathbf{M}_{R}^{(0)}\right)^{-1}\mathbf{K}_{R}^{(0)} \end{aligned} \tag{A5}$$

By using Eq. (A5), the reduced system matrices for the 0th iteration can be constructed as

$$\begin{aligned} \mathbf{K}_{R}^{(1)} &= \left[\left(\mathbf{t}_{(1)}^{(1)}\right)^{\mathrm{T}} \quad \mathbf{I} \quad \left(\mathbf{t}_{(2)}^{(1)}\right)^{\mathrm{T}}\right]\begin{bmatrix} \mathbf{K}_{ss}^{(1)} & \mathbf{K}_{sm}^{(1)} & \\ \mathbf{K}_{ms}^{(1)} & \mathbf{K}_{mm} & \mathbf{K}_{ms}^{(2)} \\ & \mathbf{K}_{sm}^{(2)} & \mathbf{K}_{ss}^{(2)} \end{bmatrix} \\ &\quad \begin{bmatrix} \mathbf{t}_{(1)}^{(1)} \\ \mathbf{I} \\ \mathbf{t}_{(2)}^{(1)} \end{bmatrix} = \left(\mathbf{t}_{(1)}^{(1)}\right)^{\mathrm{T}}\mathbf{K}_{ss}^{(1)}\mathbf{t}_{(1)}^{(1)} + \mathbf{K}_{ms}^{(1)}\mathbf{t}_{(1)}^{(1)} + \left(\mathbf{t}_{(1)}^{(1)}\right)^{\mathrm{T}}\mathbf{K}_{sm}^{(1)} \\ &\quad + \mathbf{K}_{mm} + \left(\mathbf{t}_{(2)}^{(1)}\right)^{\mathrm{T}}\mathbf{K}_{sm}^{(2)} + \mathbf{K}_{ms}^{(2)}\mathbf{t}_{(2)}^{(1)} + \left(\mathbf{t}_{(2)}^{(1)}\right)^{\mathrm{T}}\mathbf{K}_{ss}^{(2)}\mathbf{t}_{(2)}^{(1)} \\ \mathbf{M}_{R}^{(1)} &= \left[\left(\mathbf{t}_{(1)}^{(1)}\right)^{\mathrm{T}} \quad \mathbf{I} \quad \left(\mathbf{t}_{(2)}^{(1)}\right)^{\mathrm{T}}\right]\begin{bmatrix} \mathbf{M}_{ss}^{(1)} & \mathbf{M}_{sm}^{(1)} & \\ \mathbf{M}_{ms}^{(1)} & \mathbf{M}_{mm} & \mathbf{M}_{ms}^{(2)} \\ & \mathbf{M}_{sm}^{(2)} & \mathbf{M}_{ss}^{(2)} \end{bmatrix} \\ &\quad \begin{bmatrix} \mathbf{t}_{(1)}^{(1)} \\ \mathbf{I} \\ \mathbf{t}_{(2)}^{(1)} \end{bmatrix} = \left(\mathbf{t}_{(1)}^{(1)}\right)^{\mathrm{T}}\mathbf{M}_{ss}^{(1)}\mathbf{t}_{(1)}^{(1)} + \mathbf{M}_{ms}^{(1)}\mathbf{t}_{(1)}^{(1)} + \left(\mathbf{t}_{(1)}^{(1)}\right)^{\mathrm{T}}\mathbf{M}_{sm}^{(1)} \\ &\quad + \mathbf{M}_{mm} + \left(\mathbf{t}_{(2)}^{(1)}\right)^{\mathrm{T}}\mathbf{M}_{sm}^{(2)} + \mathbf{M}_{ms}^{(2)}\mathbf{t}_{(2)}^{(1)} + \left(\mathbf{t}_{(2)}^{(1)}\right)^{\mathrm{T}}\mathbf{M}_{ss}^{(2)}\mathbf{t}_{(2)}^{(1)} \end{aligned} \tag{A6}$$

From Eq. (A6), it is clear that the reduced system matrices are constructed in each subsystem and assembled into whole system. This result is equivalent to the standard IRS not iterated IRS. Thus, it is defined as the 0th iteration in this paper.

Next, for the first iteration, i.e., k=2 the reduced matrices $\mathbf{K}_{R}^{(1)}, \mathbf{M}_{R}^{(1)}$ and the transformation matrices

$\mathbf{t}_{(1)}^{(1)}, \mathbf{t}_{(2)}^{(1)}$ obtained in the previous step are used in the next construction of transformation matrices as

$$\mathbf{t}_{(1)}^{(2)} = -\left(\mathbf{K}_{ss}^{(1)}\right)^{-1}\mathbf{K}_{sm}^{(1)} + \left(\mathbf{K}_{ss}^{(1)}\right)^{-1}\left(\mathbf{M}_{ss}^{(1)}\mathbf{t}_{(1)}^{(1)} + \mathbf{M}_{sm}^{(1)}\right)\left(\mathbf{M}_{R}^{(1)}\right)^{-1}\mathbf{K}_{R}^{(1)}$$
$$\mathbf{t}_{(2)}^{(2)} = -\left(\mathbf{K}_{ss}^{(2)}\right)^{-1}\mathbf{K}_{sm}^{(2)} + \left(\mathbf{K}_{ss}^{(2)}\right)^{-1}\left(\mathbf{M}_{ss}^{(2)}\mathbf{t}_{(2)}^{(1)} + \mathbf{M}_{sm}^{(2)}\right)\left(\mathbf{M}_{R}^{(1)}\right)^{-1}\mathbf{K}_{R}^{(1)} \tag{A7}$$

These are updated transformation matrices. The reduced matrices of the first iteration are also calculated as

$$\mathbf{K}_{R}^{(2)} = \left(\mathbf{t}_{(1)}^{(2)}\right)^{T}\mathbf{K}_{ss}^{(1)}\mathbf{t}_{(1)}^{(2)} + \mathbf{K}_{ms}^{(1)}\mathbf{t}_{(1)}^{(2)} + \left(\mathbf{t}_{(1)}^{(2)}\right)^{\mathrm{T}}\mathbf{K}_{sm}^{(1)} + \mathbf{K}_{mm} + \left(\mathbf{t}_{(2)}^{(2)}\right)^{\mathrm{T}}\mathbf{K}_{sm}^{(2)} + \mathbf{K}_{ms}^{(2)}\mathbf{t}_{(2)}^{(2)} + \left(\mathbf{t}_{(2)}^{(2)}\right)^{\mathrm{T}}\mathbf{K}_{ss}^{(2)}\mathbf{t}_{(2)}^{(2)} \tag{A8}$$
$$\mathbf{M}_{R}^{(2)} = \left(\mathbf{t}_{(1)}^{(2)}\right)^{\mathrm{T}}\mathbf{M}_{ss}^{(1)}\mathbf{t}_{(1)}^{(2)} + \mathbf{M}_{ms}^{(1)}\mathbf{t}_{(1)}^{(2)} + \left(\mathbf{t}_{(1)}^{(2)}\right)^{\mathrm{T}}\mathbf{M}_{sm}^{(1)} + \mathbf{M}_{mm} + \left(\mathbf{t}_{(2)}^{(2)}\right)^{\mathrm{T}}\mathbf{M}_{sm}^{(2)} + \mathbf{M}_{ms}^{(2)}\mathbf{t}_{(2)}^{(2)} + \left(\mathbf{t}_{(2)}^{(2)}\right)^{\mathrm{T}}\mathbf{M}_{ss}^{(2)}\mathbf{t}_{(2)}^{(2)}$$

Consequently, the iterative form of transformation matrix and the reduced matrices for (k-1)th iteration are given by

$$\mathbf{T}^{(k)} = \begin{bmatrix}\mathbf{t}_{(1)}^{(k)} \\ \mathbf{I}_{mm} \\ \mathbf{t}_{(2)}^{(k)}\end{bmatrix}, \quad \mathbf{K}_{R}^{(k)} = \left(\mathbf{T}^{(k)}\right)^{\mathrm{T}}\mathbf{K}\mathbf{T}^{(k)}, \quad \mathbf{M}_{R}^{(k)} = \left(\mathbf{T}^{(k)}\right)^{\mathrm{T}}\mathbf{M}\mathbf{T}^{(k)} \tag{A9}$$

A.2 Iterative substructuring procedure for the present method

The iterative forms of Eq. (26) for k=1, 2, 3…, can be written as

$$\mathbf{t}_{(1)}^{(k)} = \mathbf{t}_{G}^{(1)} + \left(\mathbf{K}_{ss}^{(1)}\right)^{-1}\left(\mathbf{M}_{sm}^{(1)} + \mathbf{M}_{ss}^{(1)}\mathbf{t}_{(1)}^{(k-1)}\right)\left(\mathbf{M}_{d}^{(k-1)}\right)^{-1}\mathbf{K}_{G}$$
$$\mathbf{t}_{(2)}^{(k)} = \mathbf{t}_{G}^{(2)} + \left(\mathbf{K}_{ss}^{(2)}\right)^{-1}\left(\mathbf{M}_{sm}^{(2)} + \mathbf{M}_{ss}^{(2)}\mathbf{t}_{(2)}^{(k-1)}\right)\left(\mathbf{M}_{d}^{(k-1)}\right)^{-1}\mathbf{K}_{G} \tag{A10}$$

where

$$\begin{aligned}\mathbf{M}_{d}^{(k-1)} &= \left(\mathbf{M}_{d}^{(1)}\right)^{(k-1)} + \left(\mathbf{M}_{d}^{(2)}\right)^{(k-1)} \\ &= \mathbf{M}_{G}^{(1)} + \left\{\mathbf{M}_{ms}^{(1)} + \left(\mathbf{t}_{G}^{(1)}\right)^{\mathrm{T}}\mathbf{M}_{ss}^{(1)}\right\}\left(\mathbf{t}_{d}^{(1)}\right)^{(k-1)} \\ &\quad + \mathbf{M}_{G}^{(2)} + \left\{\mathbf{M}_{ms}^{(2)} + \left(\mathbf{t}_{G}^{(2)}\right)^{\mathrm{T}}\mathbf{M}_{ss}^{(2)}\right\}\left(\mathbf{t}_{d}^{(2)}\right)^{(k-1)} \\ &= \left(\mathbf{M}_{mm}^{(1)} + \mathbf{M}_{ms}^{(1)}\mathbf{t}_{(1)}^{(k-1)}\right) + \left(\mathbf{t}_{G}^{(1)}\right)^{\mathrm{T}}\left(\mathbf{M}_{sm}^{(1)} + \mathbf{M}_{ss}^{(1)}\mathbf{t}_{(1)}^{(k-1)}\right) \\ &\quad + \left(\mathbf{M}_{mm}^{(2)} + \mathbf{M}_{ms}^{(2)}\mathbf{t}_{(2)}^{(k-1)}\right) + \left(\mathbf{t}_{G}^{(2)}\right)^{\mathrm{T}}\left(\mathbf{M}_{sm}^{(2)} + \mathbf{M}_{ss}^{(2)}\mathbf{t}_{(2)}^{(k-1)}\right)\end{aligned} \tag{A11}$$

As in Eq. (A2), the initial approximations of the transformation matrices are given by

$$\mathbf{t}_{(1)}^{(0)} = -\left(\mathbf{K}_{ss}^{(1)}\right)^{-1}\mathbf{K}_{sm}^{(1)}, \quad \mathbf{t}_{G}^{(1)} = \mathbf{t}_{(1)}^{(0)}$$
$$\mathbf{t}_{(2)}^{(0)} = -\left(\mathbf{K}_{ss}^{(2)}\right)^{-1}\mathbf{K}_{sm}^{(2)}, \quad \mathbf{t}_{G}^{(2)} = \mathbf{t}_{(2)}^{(0)} \tag{A12}$$

where $\mathbf{t}_{G}^{(1)}$ and $\mathbf{t}_{G}^{(2)}$ will be used to construct the $\mathbf{M}_{\mathrm{d}}$ matrix in the iterative processing. By these transformation matrices, the Guyan stiffness reduction matrix is obtained as

$$\begin{aligned}\mathbf{K}_{Guyan} &= \left[\left(\mathbf{t}_{(1)}^{(0)}\right)^{T} \quad \mathbf{I}_{mm} \quad \left(\mathbf{t}_{(2)}^{(0)}\right)^{T}\right] \begin{bmatrix}\mathbf{K}_{ss}^{(1)} & \mathbf{K}_{sm}^{(1)} & \\ \mathbf{K}_{ms}^{(1)} & \mathbf{K}_{mm} & \mathbf{K}_{ms}^{(2)} \\ & \mathbf{K}_{sm}^{(2)} & \mathbf{K}_{ss}^{(2)}\end{bmatrix}\begin{bmatrix}\mathbf{t}_{(1)}^{(0)} \\ \mathbf{I}_{mm} \\ \mathbf{t}_{(2)}^{(0)}\end{bmatrix} \\ &= \left(\mathbf{t}_{(1)}^{(0)}\right)^{T}\mathbf{K}_{ss}^{(1)}\mathbf{t}_{(1)}^{(0)} + \mathbf{K}_{ms}^{(1)}\mathbf{t}_{(1)}^{(0)} \\ &\quad + \left(\mathbf{t}_{(1)}^{(0)}\right)^{T}\mathbf{K}_{sm}^{(1)} + \mathbf{K}_{mm} + \left(\mathbf{t}_{(2)}^{(0)}\right)^{T}\mathbf{K}_{sm}^{(2)} \\ &\quad + \mathbf{K}_{ms}^{(2)}\mathbf{t}_{(2)}^{(0)} + \left(\mathbf{t}_{(2)}^{(0)}\right)^{T}\mathbf{K}_{ss}^{(2)}\mathbf{t}_{(2)}^{(0)}\end{aligned} \tag{A13}$$

By using Eq. (A13), the starting system matrices are given by

$$\begin{aligned}\mathbf{K}_{G} &= \mathbf{K}_{Guyan} \\ \mathbf{M}_{d}^{(0)} &= \left(\mathbf{M}_{mm}^{(1)} + \mathbf{M}_{ms}^{(1)}\mathbf{t}_{(1)}^{(0)}\right) + \left(\mathbf{t}_{G}^{(1)}\right)^{\mathrm{T}} \left(\mathbf{M}_{sm}^{(1)} + \mathbf{M}_{ss}^{(1)}\mathbf{t}_{(1)}^{(0)}\right) + \left(\mathbf{M}_{mm}^{(2)} + \mathbf{M}_{ms}^{(2)}\mathbf{t}_{(2)}^{(0)}\right) \\ &\quad + \left(\mathbf{t}_{G}^{(2)}\right)^{\mathrm{T}}\left(\mathbf{M}_{sm}^{(2)} + \mathbf{M}_{ss}^{(2)}\mathbf{t}_{(2)}^{(0)}\right)\end{aligned} \tag{A14}$$

Substituting Eq. (A14) into Eq. (A10), the initial transformation matrices for k=1, i.e. 0th iteration are given by

$$\mathbf{t}_{(1)}^{(1)} = \mathbf{t}_{G}^{(1)} + \left(\mathbf{K}_{ss}^{(1)}\right)^{-1}\left(\mathbf{M}_{sm}^{(1)} + \mathbf{M}_{ss}^{(1)}\mathbf{t}_{(1)}^{(0)}\right)\left(\mathbf{M}_{d}^{(0)}\right)^{-1}\mathbf{K}_{G}$$
$$\mathbf{t}_{(2)}^{(1)} = \mathbf{t}_{G}^{(2)} + \left(\mathbf{K}_{ss}^{(2)}\right)^{-1}\left(\mathbf{M}_{sm}^{(2)} + \mathbf{M}_{ss}^{(2)}\mathbf{t}_{(2)}^{(0)}\right)\left(\mathbf{M}_{d}^{(0)}\right)^{-1}\mathbf{K}_{G} \tag{A15}$$

Substituting Eq. (A15) into Eq. (A6), the reduced system matrices of 0th iteration can be constructed as

$$\mathbf{K}_{R}^{(1)} = \left(\mathbf{t}_{(1)}^{(1)}\right)^{\mathrm{T}}\mathbf{K}_{ss}^{(1)}\mathbf{t}_{(1)}^{(1)} + \mathbf{K}_{ms}^{(1)}\mathbf{t}_{(1)}^{(1)} + \left(\mathbf{t}_{(1)}^{(1)}\right)^{\mathrm{T}}\mathbf{K}_{sm}^{(1)} + \mathbf{K}_{mm} + \left(\mathbf{t}_{(2)}^{(1)}\right)^{\mathrm{T}}\mathbf{K}_{sm}^{(2)} + \mathbf{K}_{ms}^{(2)}\mathbf{t}_{(2)}^{(1)} + \left(\mathbf{t}_{(2)}^{(1)}\right)^{\mathrm{T}}\mathbf{K}_{ss}^{(2)}\mathbf{t}_{(2)}^{(1)}$$
$$\mathbf{M}_{R}^{(1)} = \left(\mathbf{t}_{(1)}^{(1)}\right)^{\mathrm{T}}\mathbf{M}_{ss}^{(1)}\mathbf{t}_{(1)}^{(1)} + \mathbf{M}_{ms}^{(1)}\mathbf{t}_{(1)}^{(1)} + \left(\mathbf{t}_{(1)}^{(1)}\right)^{\mathrm{T}}\mathbf{M}_{sm}^{(1)} + \mathbf{M}_{mm} + \left(\mathbf{t}_{(2)}^{(1)}\right)^{\mathrm{T}}\mathbf{M}_{sm}^{(2)} + \mathbf{M}_{ms}^{(2)}\mathbf{t}_{(2)}^{(1)} + \left(\mathbf{t}_{(2)}^{(1)}\right)^{\mathrm{T}}\mathbf{M}_{ss}^{(2)}\mathbf{t}_{(2)}^{(1)} \tag{A16}$$

This result is the same as that of Eq. (A6). And they are equivalent to the standard IRS method.

Next, for the first iteration, i.e., k=2 the $\mathbf{M}_d^{(1)}$ matrix can be constructed by using Eq. (A16).

$$\begin{aligned}\mathbf{M}_d^{(1)} &= \left(\mathbf{M}_{mm}^{(1)} + \mathbf{M}_{ms}^{(1)}\mathbf{t}_{(1)}^{(1)}\right) + \left(\mathbf{t}_G^{(1)}\right)^{\mathrm{T}} \\ &\left(\mathbf{M}_{sm}^{(1)} + \mathbf{M}_{ss}^{(1)}\mathbf{t}_{(1)}^{(1)}\right) + \left(\mathbf{M}_{mm}^{(2)} + \mathbf{M}_{ms}^{(2)}\mathbf{t}_{(2)}^{(1)}\right) \\ &+ \left(\mathbf{t}_G^{(2)}\right)^{\mathrm{T}}\left(\mathbf{M}_{sm}^{(2)} + \mathbf{M}_{ss}^{(2)}\mathbf{t}_{(2)}^{(1)}\right)\end{aligned} \tag{A17}$$

From Eq. (A17), the transformation matrices for the first iteration can be calculated as

$$\begin{aligned}\mathbf{t}_{(1)}^{(2)} &= \mathbf{t}_G^{(1)} + \left(\mathbf{K}_{ss}^{(1)}\right)^{-1}\left(\mathbf{M}_{sm}^{(1)} + \mathbf{M}_{ss}^{(1)}\mathbf{t}_{(1)}^{(1)}\right)\left(\mathbf{M}_d^{(1)}\right)^{-1}\mathbf{K}_G \\ \mathbf{t}_{(2)}^{(2)} &= \mathbf{t}_G^{(2)} + \left(\mathbf{K}_{ss}^{(2)}\right)^{-1}\left(\mathbf{M}_{sm}^{(2)} + \mathbf{M}_{ss}^{(2)}\mathbf{t}_{(2)}^{(1)}\right)\left(\mathbf{M}_d^{(1)}\right)^{-1}\mathbf{K}_G\end{aligned} \tag{A18}$$

In the same manner, the system reduced matrices of the first iteration are given by

$$\begin{aligned}\mathbf{K}_R^{(2)} &= \left(\mathbf{t}_{(1)}^{(2)}\right)^{\mathrm{T}}\mathbf{K}_{ss}^{(1)}\mathbf{t}_{(1)}^{(2)} + \mathbf{K}_{ms}^{(1)}\mathbf{t}_{(1)}^{(2)} \\ &+ \left(\mathbf{t}_{(1)}^{(2)}\right)^{\mathrm{T}}\mathbf{K}_{sm}^{(1)} + \mathbf{K}_{mm} + \left(\mathbf{t}_{(2)}^{(2)}\right)^{\mathrm{T}}\mathbf{K}_{sm}^{(2)} \\ &+ \mathbf{K}_{ms}^{(2)}\mathbf{t}_{(2)}^{(2)} + \left(\mathbf{t}_{(2)}^{(2)}\right)^{\mathrm{T}}\mathbf{K}_{ss}^{(2)}\mathbf{t}_{(2)}^{(2)} \\ \mathbf{M}_R^{(2)} &= \left(\mathbf{t}_{(1)}^{(2)}\right)^{\mathrm{T}}\mathbf{M}_{ss}^{(1)}\mathbf{t}_{(1)}^{(2)} + \mathbf{M}_{ms}^{(1)}\mathbf{t}_{(1)}^{(2)} \\ &+ \left(\mathbf{t}_{(1)}^{(2)}\right)^{\mathrm{T}}\mathbf{M}_{sm}^{(1)} + \mathbf{M}_{mm} + \left(\mathbf{t}_{(2)}^{(2)}\right)^{\mathrm{T}}\mathbf{M}_{sm}^{(2)} \\ &+ \mathbf{M}_{ms}^{(2)}\mathbf{t}_{(2)}^{(2)} + \left(\mathbf{t}_{(2)}^{(2)}\right)^{\mathrm{T}}\mathbf{M}_{ss}^{(2)}\mathbf{t}_{(2)}^{(2)}\end{aligned} \tag{A19}$$

Consequently, the modified forms of iterative transformation matrix and the reduced matrices are

$$\mathbf{T}^{(k)} = \begin{bmatrix}\mathbf{t}_{(1)}^{(k)} \\ \mathbf{I}_{mm} \\ \mathbf{t}_{(2)}^{(k)}\end{bmatrix}, \quad \mathbf{K}_R^{(k)} = \left(\mathbf{T}^{(k)}\right)^{\mathrm{T}}\mathbf{K}\mathbf{T}^{(k)}, \quad \mathbf{M}_R^{(k)} = \left(\mathbf{T}^{(k)}\right)^{\mathrm{T}}\mathbf{M}\mathbf{T}^{(k)} \tag{A20}$$